backward, the histories of both C_k and d_k are obtained by virtue of Eq. (7). Since A and B and S_f are symmetrical, S_k is clearly symmetrical for any k. Equating the coefficients of x_i in Eq. (8), we obtain

$$\Psi_i = (-f_r^TA)_i + [(df_d + g)^TBC]_i + \{(F - GC)^TS_{k+1}[(I - Gd)f_d - Gg]\}_i + [\Psi_{k+1}(F - GC)]_i \tag{11}$$

The end conditions of Eq. (11) are obtained by equating Eqs. (2) and (4) for the N stage. This yields

$$\Psi(N) = -f_{r,N}^TS_f \tag{12}$$

With the histories of S_k, C_k, d_k already precomputed, and with g_k depending on Ψ_{k+1} and S_{k+1} by use of Eq. (7), Eq. (11) can now be propagated backward from Eq. (12) to yield nontrivial solutions for both Ψ_k and g_k. This completes the precomputations for the optimal control, Eq. (6). With a full, nonanticipative accessibility of the state, this control law consisting of two parts, can now be implemented. The first part is linear in the current state, with the gains C_k stored in memory, and thus has the form of feedback. The second part consists of precomputed functions of f_d and f_r that are prescribed; thus it can be viewed as a feed-forward term. Equation (8) forces a third regression, in the parameter Φ, that has a nontrivial solution for $Q \neq 0$. This equation is omitted from the present discussion since the optimal control is independent of w, as expected in the linear process with zero-mean noise. The existence of nontrivial solutions for the coefficients of Eq. (8) validates the proposed solution, Eq. (4), for the minimum cost functional.

IV. Alternative Formulation

It may be convenient at times to use the relative state vector X_r in the tracking problem formulation, where X_r is defined by

$$X_r(k) \triangleq X(k) - f_r(k) \tag{13}$$

Equation (1) is then equivalently described by difference equations in the relative state vector, that is

$$X_r(k+1) = F_kX_r(k) + G_kU(k) + f_d(k) + F_kf_r(k) - f_r(k+1) + W(k) \tag{14}$$

To minimize the cost in the relative coordinates J_r, we write

$$J_r = E\left\{\frac{1}{2}X_r^T(N)S_fX_r(N) + \sum_{k=1}^{N-1}\left[\frac{1}{2}X_r^T(k)A_kX_r(k) + \frac{1}{2}U^T(k)B_kU(k)\right]\Big|X_r(k)\right\} \tag{15}$$

The analysis in Secs. II and III is clearly valid for this formulation if we replace $f_d(k)$ everwhere by the expression

$$f_d(k) + F_kf_r(k) - f_r(k+1) \triangleq f_{d,r} \tag{16}$$

and cancel the term $(-f_rA)_i$ in Eq. (11).

The end conditions for the regressions of both Ψ in Eq. (11) and Φ will be zero for this case but their solutions will clearly remain nontrivial. The consequent optimal control will surely remain in the form of Eq. (6), but with $f_{d,r}$ replacing f_d and X_r replacing X, which often leads to a simpler implementation.

V. Concluding Remarks

Tracking requirements and prescribed inputs both lead to the same structure of the cost functional when they appear in linear-quadratic optimization problems. This remains true in the presence of a zero-mean process noise, enabling one to address the combination of these three classical elements in a unified and inclusive approach. Applying dynamic programming to the discrete version of the problem, we readily obtain a closed-form solution for the optimal control, which consists of two parts. The first part is the classical optimal feedback control. The second part contains two feedforward terms that are precomputed, based on the prescribed disturbance and reference.

References

[1]Garber, V., "Optimum Intercept Laws for Accelerating Targets," *AIAA Journal*, Vol. 6, No. 11, Nov. 1968, pp. 2196–2198.

[2]Bellman, R., *Dynamic Programming*, Princeton University Press, Princeton, NJ, 1957.

Molniya Orbits Obtained by the Two-Burn Method

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Introduction

SOME U.S. satellite missions require the utilization of a 12- or 24-h (period) Molniya orbit. Its inclination angle of 63.4 deg precludes rotation of the line of apsides, thus assuring that the majority of orbit time is spent in the northern or southern hemisphere, depending on where the perigee is "anchored." Launch restrictions from both ETR and WTR permit inclinations that are approximately 6½ deg above or below 63.4 deg, thus requiring a "dog-leg" maneuver, which is costly in ΔV.

Starting from a low-altitude circular orbit, a Molniya orbit can be achieved by a sequential three-burn method (adjusting apogee, perigee, and orbit inclination) or by a two-burn method, in which the first perigee burn raises apogee, and the second burn, at a true anomaly greater than 90 deg, simultaneously raises apogee and perigee to their final values, while making the required inclination change.

The purposes of this Note are threefold: 1) to demonstrate that the two-burn method is significantly more efficient than a three-burn method and is close to being the optimal solution for the cases studied; 2) to present some results that illustrate this and also permit determination of ΔVs for preliminary mission design purposes; and 3) to indicate the logic used in developing the program for IBM PC execution.

Discussion

We will assume an ELV or STS launch into a circular parking orbit. A typical ETR shuttle parking orbit is circular, at 150 n.mi. altitude, with an inclination of, at most, 57 deg. A Molniya orbit has an inclination of 63.4 deg and is highly elliptical (a typical perigee is 400 n.mi., which means, for a 12-h period, an apogee of 21,449 n.mi.) and has a perigee at the most southerly latitude.

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To transfer an ETR-launched spacecraft into a Molniya orbit, apogee, perigee, and inclination must all be raised. The most straightforward way of doing this, which we call the simplified three-burn method, is to change each element in a separate burn. The most efficient order in which to do this is to raise apogee to its final value, then raise perigee, and finally change inclination. This order is always more efficient than the other sequences, at least for normal orbital parameters. This method is not, however, the most efficient method possible. The maneuver can be accomplished in only two burns with a considerably smaller total ΔV expenditure for small (< 10–15 deg) inclination changes. Table 1 shows typical ΔV comparisons.

In the two-burn method, the first burn is made at the same place as in the three-burn method: 90 deg from the line of nodes (i.e., 0 deg true anomaly), at the perigee of the new orbit. The apogee of the new orbit is not, however, the same as the desired final apogee; it is around 70–90% of its final value depending on initial inclination and the final desired perigee altitude. The second burn, made in the other hemisphere, brings both apogee and perigee to their final values, raises the inclination to 63.4 deg, and rotates the line of nodes (Fig. 1).

The problem is to find, for given initial (e.g., parking) and final orbits, the optimum transfer orbit—i.e., the set of two burns that

Table 1 ΔV Comparison for simplified three-burn vs two-burn method, for initial perigee of 150 nmi at 57 deg inclination

Final perigee (nmi)	12-h 3-Burn ΔV_t [a]	12-h 2-Burn ΔV_t [a]	24-h 3-Burn ΔV_t [a]	24-h 2-Burn ΔV_t [a]
200	10,879.3	9,561.3	11,800.7	10,213.2
300	10,889.4	9,529.6	11,795.7	10,194.8
400	10,900.2	9,508.1	11,791.8	10,182.7
500	10,911.8	9,496.4	11,788.8	10,176.6
600	10,924.1	9,493.5	11,786.7	10,175.8
700	10,936.9	9,498.1	11,785.5	10,179.5
800	10,950.3	9,508.6	11,785.0	10,186.8
900	10,964.3	9,523.7	11,785.3	10,197.1

[a]Velocities in ft/s.

leads to the smallest total ΔV. It turns out that the problem can be most easily posed (at least for a small computer) in terms of two independent variables: the rotation of the line of nodes that results from the second burn and the true anomaly (in the transfer orbit) at which it occurs. The total ΔV required can be computed using the following relations.

The angle Δi between the orbital planes of the transfer and final orbits can be computed from the spherical law of cosines (see Fig. 2a)

$$\cos \Delta i = -\cos i_0 \cos(180\ \text{deg} - i) + \sin i_0 \sin(180\ \text{deg} - i)\cos\Delta\Omega$$

where i_0 is the initial inclination (say, 57 deg), i the final inclination (63.4 deg), and $\Delta\Omega$ the first of the two independent variables, the rotation of the line of nodes caused by the second burn. From the spherical law of sines (and the fact that the argument of perigee of a Molniya orbit is always -90 deg),

$$\theta_f = 90\ \text{deg} + \sin^{-1}[\sin i_0(\sin\Delta\Omega/\sin\Delta i)]$$

where θ_f is the true anomaly in the final orbit of the second burn.

If P is the period of the final orbit (usually 12 or 24 h) and r_p is its radius at perigee, then for this orbit

$$a_f = (\mu P^2/4\pi^2)^{1/3} \qquad \text{and} \qquad e_f = (a_q - r_p)/a_q$$

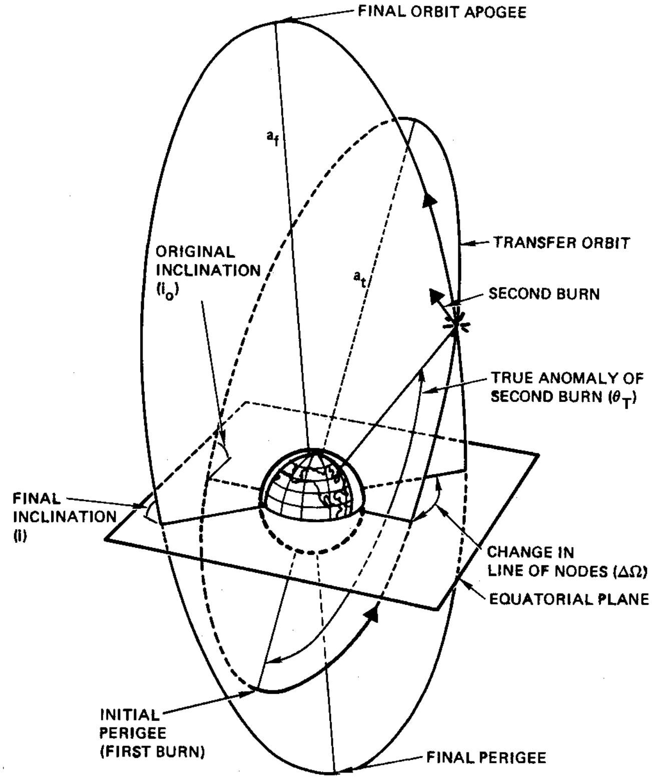

Fig. 1 Geometry of two-burn Molniya orbit achievement.

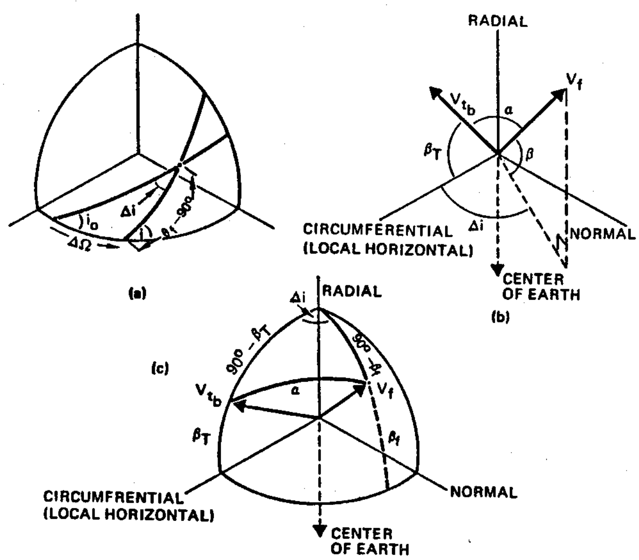

Fig. 2 Spherical angle relationship.

where a_f is the semi-major axis and e_f is the eccentricity, and the radius at the second burn is

$$r = a_f(1 - e_f^2)/(1 + e_f \cos\theta_f)$$

For the transfer orbit (result of first burn), we have the similar relations

$$e_T = (a_T - R_0)/a_T \qquad \text{and} \qquad r = a_T(1 - e_T^2)/(1 + e_T \cos\theta_T)$$

where R_0 is the radius of the initial parking orbit (say, 150 nmi) and θ_T is the second of the two independent variables, the true anomaly (in the transfer orbit) of the second burn. From these, we derive that

$$a_T = R_0\ (r \cos\theta_T - R_0)/[(1 + \cos\theta_T)r - 2R_0]$$

Table 2 Δ*V*s and firing angles for 12- and 24-h Molniya orbits with initial perigee at 150 n.mi. at 57 deg inclination

Final orbit perigee (nmi)	Total delta *V* (fps)	First burn	Second burn	a_T/a_f	True anomaly (θ_T)	At second burn rotation ($\Delta\Omega$) of line of nodes	Firing angles[a] (γ)	(ψ)
				12-h Orbit				
200.0	9,561.3	7,989.6	1,571.7	0.92	132.2	6.5	70.0	89.6
300.0	9,529.6	7,950.0	1,579.6	0.91	131.6	6.0	70.4	84.4
400.0	9,508.1	7,911.2	1,596.9	0.89	131.4	5.8	70.7	79.4
500.0	9,496.4	7,878.6	1,617.8	0.88	132.0	5.6	71.0	74.6
600.0	9,493.5	7,853.5	1,640.1	0.88	133.0	5.6	71.5	70.9
700.0	9,498.1	7,827.8	1,670.3	0.87	134.1	5.6	71.6	67.2
800.0	9,508.6	7,815.7	1,692.9	0.86	135.7	5.8	72.1	64.3
900.0	9,523.7	7,802.4	1,721.4	0.86	137.1	5.9	72.5	61.6
				24-h Orbit				
200.0	10,213.2	8,950.3	1,262.8	0.92	140.9	9.3	72.6	92.0
300.0	10,194.8	8,934.2	1,260.6	0.91	140.8	9.0	72.8	87.7
400.0	10,182.7	8,918.8	1,264.0	0.91	141.0	8.8	73.0	83.5
500.0	10,176.6	8,906.5	1,270.1	0.90	141.5	8.7	73.3	79.7
600.0	10,175.8	8,896.2	1,279.6	0.89	142.2	8.7	73.6	76.2
700.0	10,179.5	8,888.3	1,291.2	0.89	143.1	8.8	73.9	73.1
800.0	10,186.8	8,880.5	1,306.3	0.88	144.0	8.9	74.1	70.2
900.0	10,197.1	8,876.0	1,321.1	0.88	145.1	9.1	74.3	67.8

[a]See Fig. 3.

This determines how large the transfer orbit is in relation to the final orbit. (We found the optimum value of a_T/a_f to be in the range of 0.86 to 0.92 for a range of different initial and final orbits; see Table 2 later).

The relevant velocities can now be found, using the general formula

$$V = \sqrt{[\mu(2/r - 1/a)]}$$

If V_0 is the initial orbital velocity, V_{Tp} and V_{Tb} are the velocity magnitudes in the transfer orbit at perigee (first burn) and the second burn, respectively, and V_f is the magnitude of the velocity after the second burn,

$$V_0 = \sqrt{(\mu/R_0)}, \qquad V_{Tp} = \sqrt{[\mu(2/R_0 - 1/a_T)]}$$

$$V_{Tb} = \sqrt{[\mu(2/r - 1/a_T)]}, \qquad V_f = \sqrt{[\mu(2/r - 1/a)]}$$

From this, we can immediately calculate the ΔV for the first burn, $\Delta V_1 = V_{Tp} - V_0$.

To find the ΔV for the second burn, we need the flight-path angle (angle with the local horizontal) in each orbit, found by the formula

$$\beta = \tan^{-1}[e \sin\theta/(1 + e \cos\theta)]$$

Then, if α is the angle between the velocity vectors V_{Tb} and V_f (Fig. 2b, c),

$$\cos\alpha = \cos(90\text{ deg} - \beta_T)\cos(90\text{ deg} - \beta_f) + \sin(90\text{ deg} - \beta_T)\sin(90\text{ deg} - \beta_f)\cos\Delta i$$

and by the usual (plane trigonometry) law of cosines,

$$\Delta V_2 = \sqrt{(V_{Tb}{}^2 + V_f{}^2 - 2V_{Tb}V_f \cos\alpha)}$$

The total change in velocity ΔV_{total}, is, of course, the sum of ΔV_1 and ΔV_2.

We also used the foregoing information to find the firing angles. First, we found the components of V_{Tb} and V_f in the coordinate system of Fig. 2b

$$V_{Tb} = (V_{Tb} \cos\beta_T,\ 0,\ V_{Tb} \sin\beta_T)$$

$$V_f = (V_f \cos\beta_f \cos\Delta i,\ V_f \cos\beta_f \sin\Delta i, V_f \sin\beta_f)$$

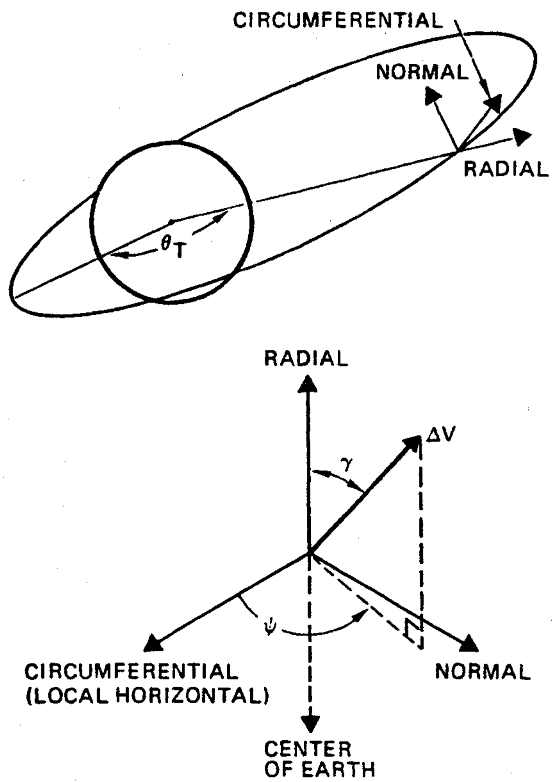

Fig. 3 Firing angles at second-burn true anomaly.

Thus, in this system;

$$\Delta V_2 = (V_f \cos\beta_f \cos\Delta i - V_{Tb} \cos\beta_T,\ V_f \cos\beta \sin\Delta i,\ V_f \sin\beta_f - V_{Tb} \sin\beta_T)$$

which, if converted to spherical coordinates, gives the firing angles γ and ψ (Fig. 3).

At this point, any two-variable optimization method can be used to find the minimum of the function ΔV ($\Delta\Omega$, θ_T). We tried various numerical methods (using an IBM PC-XT) and found that a slightly modified conjugate gradient method using a numerical approximation of the partial derivatives worked relatively well, and took about 6 min (on the average) to converge. In any case, we did succeed in finding the optimum two-burn solution for a range of final initial orbits—final orbits with periods of 12 and 24 h and perigee altitudes of 200–900 n.mi. and initial orbits with altitude 150 n.mi. and inclinations of 45 and 57 deg. Table 2 shows the optimal solutions we calculated for the various values of P and r_p for 57 deg initial inclination—the total ΔV(broken into ΔV_1, ΔV_2), the size of the transfer orbit (a_T/a_f), the values of the two independent variables ($\Delta\Omega$, θ_T), and the firing angles (γ, ψ). For an initial inclination of 45 deg, the ΔVs_{total} (for the range of perigee attitudes) are of the order of 15,600 ft/s for a 12-h orbit and around 1000 ft/s higher for a 24-h orbit.

The results show, first of all, that the two-burn method always requires a significantly smaller ΔV than the simplified three-burn (as was shown in Table 1). It is also obvious that the minimum total ΔV depends strongly on the initial inclination. The dependence on the final perigee altitude, on the other hand, is very weak—a difference of 100 n.mi. changes the total ΔV by only a few feet per second.

Conclusions

It is clear that within the parameters studied, the two-burn method is considerably more efficient than the simplified, step-by-step three-burn method. While the ΔV can be reduced further by making a small plane change at the first (perigee) burn or by using an optimized three-burn solution,[1] our two-burn approximation (which is easily implemented on a small computer) is accurate enough for preliminary design purposes, and essentially optimum for plane changes less than 10 deg.

Acknowledgment

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Reference

Betts, J. T., "Optimal Three-Burn Orbit Transfer," *AIAA Journal*, Vol. 15, June 1977, pp. 861–864.

Optimal Proportional Navigation

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Introduction

UNTIL now, the investigation of optimal guidance laws has been restricted to the purely proportional type,[1,2] wherein the commanded accelerations are applied normal to the missile velocity and both missile and target are assumed to have constant speeds. Because of the inherent differences in the equations of motion, these results cannot be applied normal to the line of sight (LOS). Under true proportional navigation (TPN), the commanded missile acceleration applied normal to the LOS, a_{M_c}, is equal to $u\dot{\theta}$, where u is a proportionality constant and $\dot{\theta}$ the LOS rate. Using the exact nonlinear equations of motion,[3,4] it has been shown that the time of capture is reduced as u increases; however, the control expenditure also increases at the same time. A reasonable compromise seems to be possible if a quadratic performance index is used to optimize a weighted combination of the time of capture and the expenditure of maneuvering energy. Based on the exact nonlinear equations of motion in the plane, an optimal trajectory of u for a vehicle pursuing a maneuvering target is derived in this paper. It is assumed that a complete knowledge of the target's motion is available to the missile.

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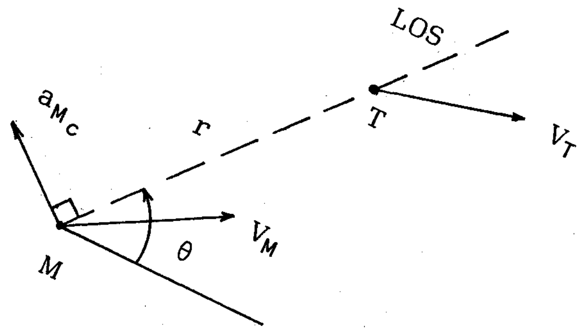

Fig. 1 Planar pursuit geometry.

Problem Formulation

A missile M is attempting to capture a target T as shown in Fig. 1. The vehicles have velocity components V_{M_r}, $V_{M\theta}$, V_{T_r}, and $V_{T\theta}$, respectively, in the polar coordinate system (r, θ) with the origin fixed at the missile. The equations of motion are

$$\ddot{r} - r\dot{\theta}^2 = \dot{V}_{T_r} \qquad r(0) = r_0 \tag{1a}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \dot{V}_{T\theta} - u\dot{\theta} \qquad \theta(0) = \theta_0 \tag{1b}$$

where r is the range of the target and θ the aspect angle measured with respect to an inertial reference. Equivalently, we have

$$\dot{V}_r = V_\theta\dot{\theta} + \dot{V}_{T_r} \qquad V_r(0) = V_{r_0} \tag{2a}$$

$$\dot{V}_\theta = -V_r\dot{\theta} + \dot{V}_{T\theta} - u\dot{\theta} \qquad V_\theta(0) = V_{\theta 0} \tag{2b}$$

where $V_r = \dot{r} = V_{T_r} - V_{M_r}$ and $V_\theta = r\dot{\theta} = V_{T\theta} - V_{M\theta}$ are the components of the target velocity relative to the missile. By changing the independent variable from t to θ, Eqs. (2) are reduced to the simple form

$$V_r' = V_\theta + V_{T_r}' \qquad V_r(\theta_0) = V_{r_0} \tag{3a}$$

$$V_\theta' = -V_r + V_{T\theta}' - u \qquad V_\theta(\theta_0) = V_{\theta_0} \tag{3b}$$

where V_{T_r}' and $V_{T\theta}'$ are given continuous functions of θ and primes denote differentiation with respect to θ.

It has been shown[5] that θ is a monotonically increasing or decreasing function of t, depending on the sign of V_{θ_0} (we will assume a positive V_{θ_0} in the following discussion). Thus, the problem to be solved is to find a control u such that the capture point $r(\theta_f) = 0$ is assured while the performance index

$$J = \theta_f + \rho\int_{\theta_0}^{\theta_f} u^2 \, d\theta$$

is minimized. The weighting factor ρ is selected in such a way that the required acceleration is realistic. The range of the parameter ρ and the initial conditions (V_{r_0}, V_{θ_0}) that ensure the